

PBW DEFORMATIONS OF A FOMIN–KIRILLOV ALGEBRA AND OTHER EXAMPLES

I. HECKENBERGER AND L. VENDRAMIN

ABSTRACT. We begin the study of PBW deformations of graded algebras relevant to the theory of Hopf algebras. One of our examples is the Fomin–Kirillov algebra \mathcal{E}_3 . Another one appeared in a paper of García Iglesias and Vay. As a consequence of our methods, we determine when the deformations are semisimple and we are able to produce PBW bases and polynomial identities for these deformations.

Keywords: Clifford algebra, Fomin–Kirillov algebra, Hopf algebra, Nichols algebra, PBW deformation, polynomial identity.

INTRODUCTION

Deformations of several algebraic structures have been of great interest in the last years. Such deformations include group algebras, Lie algebras, Weyl algebras, rational Cherednik algebras, Hecke algebras and generalizations. A deformation of a graded algebra A given by generators a_1, \dots, a_n and homogeneous relations r_1, \dots, r_m is an algebra D given by generators a_1, \dots, a_n and relations $r_1 + t_1, \dots, r_m + t_m$, where each t_j is a possibly non-homogeneous element of degree lesser than the degree of r_j .

The classical Poincaré–Birkhoff–Witt Theorem for Lie algebras motivates the study of a particular family of deformations. A PBW deformation of a graded algebra A is a deformation D of A such that the associated graded algebra of D is isomorphic to A . PBW deformations have been considered in many different contexts. In the case of quadratic algebras, these deformations have been recently studied in [6, 8, 13, 23]. For N -homogeneous algebras they have been studied in [11, 4]. In the context of Hopf algebras and their actions, a related class of algebras was studied in [27] and in [26]. PBW deformations satisfying additional properties also appear in several papers where the classification of finite-dimensional pointed Hopf algebras is considered, see for example [14].

Nichols algebras over non-abelian groups form a particularly interesting family of graded algebras where not very much is known. In particular, in general they are not N -Koszul and are not even generated by homogeneous

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relations of constant degree. Nevertheless they have remarkable Hilbert series which indicates a rich internal structure. Many of the crucial results on these algebras can only be recovered by extensive Gröbner basis calculations. Nevertheless, these algebras are very important in particular since they appear as an essential tool in the classification of pointed Hopf algebras with non-abelian coradical [1], in combinatorics [3, 5, 12, 17, 20] and in mathematical physics [16, 18, 19, 22].

In this paper we begin the study of PBW deformations of some finite-dimensional Nichols algebras over non-abelian groups. Our long-term objective is to understand the structure of such Nichols algebras by means of PBW deformations. For that purpose we concentrate first on two small examples: 1) The Fomin–Kirillov algebra \mathcal{E}_3 , and 2) the Nichols algebra associated with the vertices of the tetrahedron and constant cocycle -1 .

Some particular deformations of the Fomin–Kirillov algebra \mathcal{E}_3 have been considered in [19, 18] and in [24]. The cohomology of \mathcal{E}_3 has been recently computed in [7].

We compute all PBW deformations of these Nichols algebras. It turns out that the moduli space of PBW deformations is an affine space of very small dimension. Moreover, generically the deformations are semisimple and the non-semisimple locus is determined. Our result has some applications: a) we get a PBW basis of the algebras; b) we produce non-trivial polynomial identities for our Nichols algebras that were not known before. For example, as a corollary we prove that the Fomin–Kirillov algebra \mathcal{E}_3 satisfies the *Hall identity*

$$[[x, y]^2, z] = 0.$$

The PBW deformations we find have also been obtained by García Iglesias and Vay [22]. These algebras are useful to obtain the classification of some finite-dimensional pointed Hopf algebras with non-abelian coradical. We provide a simplification in the presentation of the 72-dimensional example.

The paper is organized as follows. In Section 1 we define our algebras as certain PBW deformations of Nichols algebras compatible with the group action and we prove some basic properties. Then we recall a basic result on the simplicity of Clifford algebras that will be useful to study our examples. In Section 2 we study PBW deformations of the Fomin–Kirillov algebra \mathcal{E}_3 . In Theorem 2.10 we precisely determine when the deformations of \mathcal{E}_3 are semisimple; it is remarkable that in the cases where the deformation is not semisimple one finds either the preprojective algebra of type A_2 or the coinvariant algebra appearing in Schubert calculus [3]. In Theorem 2.11 a PBW basis is constructed and PBW deformations of \mathcal{E}_3 are classified in Theorem 2.13. In Section 4 PBW deformations of the 72-dimensional Nichols algebra associated with the vertices of the tetrahedron are studied. Using Ore extensions we produce a PBW basis for the deformation of this Nichols algebra, see Proposition 4.8, Theorems 4.9 and 4.12. In Theorem 4.10 we determine when this deformations are semisimple. Most of the results on

this deformation are based on calculations related to an intermediate algebra studied in Section 3.

We end the introduction by formulating two problems.

Problem. *Classify the PBW deformations of the known finite-dimensional Nichols algebras over groups. Decide when these algebras are semisimple.*

Problem. *Classify the PBW deformations of the Fomin–Kirillov algebras and their subalgebras appearing in [5]. Decide when these deformations are semisimple.*

Problem. *Study the representation theory of the PBW deformations of the algebras in the first two problems.*

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1. PRELIMINARIES

Thorough this paper we assume that \mathbb{K} is an algebraically closed field of characteristic $\neq 2$.

1.1. Deformations. Let \mathcal{C} be a monoidal category. (In our case, \mathcal{C} will be the category of $\mathbb{K}G$ -modules for some field \mathbb{K} and some finite group G .)

Definition 1.1. *Let A be a \mathbb{N}_0 -graded algebra in \mathcal{C} . A PBW deformation of A in \mathcal{C} is an \mathbb{N}_0 -filtered algebra D_A in \mathcal{C} such that $\text{gr } D_A \cong A$.*

The following proposition is well-known and the proof is elementary. Similar arguments have been used by Etingof and Rains for example in [10, §2.2] and [9, Theorem 6.1].

Proposition 1.2. *Let A be an \mathbb{N}_0 -graded finite-dimensional algebra. Let $r, d \in \mathbb{N}$ and let $A(\lambda)$, $\lambda \in \mathbb{K}^r$, be a family of deformations of $A = A(0)$ such that $\dim A(\lambda) \leq d$ for all $\lambda \in \mathbb{K}^r$. Assume that $\dim A(\lambda) = d$ for all λ in a Zariski dense subset of \mathbb{K}^r . Then $A(\lambda)$ is a PBW deformation of A for all $\lambda \in \mathbb{K}^r$.*

1.2. Clifford algebras. Let V be a finite-dimensional vector space with basis v_1, \dots, v_n , where $n = \dim V$, and let q be a quadratic form on V . The pair (V, q) is called a *quadratic space*. The *Clifford algebra* $C(V, q)$ is the algebra given by generators v_1, \dots, v_n and relations

$$v_i^2 = q(v_i), \quad v_j v_k + v_k v_j = q_{jk},$$

for $1 \leq i, j, k \leq n$ with $j < k$, where $q_{jk} = \frac{1}{2}(q(v_j + v_k) - q(v_j) - q(v_k))$. It is known that $\dim C(V, q) = 2^n$, see for example [25, §9.2, Corollary 2.7].

Theorem 1.3. *Let (V, q) be a quadratic space.*

- (1) *The radical of $C(V, q)$ is generated by the radical of the bilinear form B_q associated with q .*

- (2) If $\dim V$ is even and q is nondegenerate, then $C(V, q)$ is simple.
- (3) If $\dim V$ is odd and q is nondegenerate, then $C(V, q)$ is the product of two simple ideals of dimension $2^{\dim V - 1}$ each.

Proof. The claims (2) and (3) follow from [25, §9.2, Theorem 2.10]. Regarding (1), note that for any element v in the radical R of B_q , the left ideal generated by v is a nilpotent two-sided ideal of $C(V, q)$. Hence

$$C(V, q)R \subseteq \text{Rad } C(V, q).$$

Let q' be the quadratic form of V/R induced by q . Then $C(V/R, q')$ is semisimple by (2) and (3) and $C(V, q)/C(V, q)R \simeq C(V/R, q')$. Hence $\text{Rad } C(V, q) = C(V, q)R$. \square

2. THE FOMIN–KIRILLOV ALGEBRA \mathcal{E}_3

The Fomin–Kirillov algebra \mathcal{E}_3 is defined by generators a, b, c and relations

$$\begin{aligned} a^2 &= b^2 = c^2 = 0, \\ ca + bc + ab &= cb + ba + ac = 0. \end{aligned}$$

It is known that $\dim \mathcal{E}_3 = 12$. A basis is given by

$$1, a, b, c, ab, ac, ba, bc, aba, abc, bac, abac.$$

We put this algebra into a different context without using this information.

Remark 2.1. It is known that \mathcal{E}_3 is a Nichols algebra. This was first proved by Milinski and Schneider [21].

The Fomin–Kirillov algebra \mathcal{E}_3 first appeared in [12] to provide a combinatorial tool to study the structure constants of Schubert polynomials, and independently in the paper [21] of Milinski and Schneider, where pointed Hopf algebras with non-abelian coradical were studied. It also appeared in the work of Majid and Raineri [19], where applications to physics were considered. The cohomology of \mathcal{E}_3 was computed by Ştefan and Vay in [7].

Definition 2.2. For any $\alpha_1, \alpha_2 \in \mathbb{K}$ let $\mathcal{D}_3(\alpha_1, \alpha_2)$ be the deformation of \mathcal{E}_3 given by generators a, b, c and relations

$$\begin{aligned} a^2 - \alpha_1 &= b^2 - \alpha_1 = c^2 - \alpha_1 = 0, \\ ca + bc + ab - \alpha_2 &= cb + ba + ac - \alpha_2 = 0. \end{aligned}$$

Remark 2.3. A direct calculation shows that a Gröbner basis for the ideal of relations is given by

$$\begin{aligned} a^2 - \alpha_1 &= 0, & b^2 - \alpha_1 &= 0, & ca + bc + ab - \alpha_2 &= 0, \\ cb + ba + ac - \alpha_2 &= 0, & c^2 - \alpha_1 &= 0, & bab - aba - \alpha_2 b + \alpha_2 a &= 0. \end{aligned}$$

We will not use this Gröbner basis for our arguments.

Remark 2.4. For any $\alpha_1, \alpha_2 \in \mathbb{K}$ the algebra $\mathcal{D}_3(\alpha_1, \alpha_2)$ is a \mathbb{S}_3 -module algebra, where

$$\begin{aligned} (12) \cdot a &= -b, & (12) \cdot b &= -a, & (12) \cdot c &= -c, \\ (23) \cdot a &= -a, & (23) \cdot b &= -c, & (23) \cdot c &= -b. \end{aligned}$$

Lemma 2.5. *Let $u = a - b$, $v = b - c$ and $w = c - a$ in $\mathcal{D}_3(\alpha_1, \alpha_2)$. Then*

$$\begin{aligned} ua &= -bu, & ub &= -au, & uc &= (c - a - b)u, \\ uv &= vu, & uw &= wu, & vw &= wv. \end{aligned}$$

Moreover

$$uv + vw + uw = \alpha_2 - 3\alpha_1, \quad u^3 = (3\alpha_1 - \alpha_2)u, \quad u^2v + uv^2 = 0, \quad uvw = 0.$$

Proof. The first formula is obtained as follows:

$$ua = (a - b)a = a^2 - ba = b^2 - ba = b(b - a) = -bu.$$

By acting with $(12) \in \mathbb{S}_3$ we obtain that $ub = -au$. Now

$$\begin{aligned} (c - a - b)u &= (c - a - b)(a - b) = ca - cb - \alpha_1 + ab - ba + \alpha_1 \\ &= ca + ab - (cb + ba) = -bc + \alpha_2 + ac - \alpha_2 = (a - b)c = uc. \end{aligned}$$

We conclude that $uv = vu$ and $uw = wu$. By acting with (12) on $uv = vu$ we obtain that $vw = wv$.

From the definitions of u , v and w it follows that

$$uv + vw + uw = vw - u^2 = \alpha_2 - 3\alpha_1.$$

From $cb + ba + ac - \alpha_2 = 0$ one obtains that

$$bcb + \alpha_1 a + bac - \alpha_2 b = \alpha_1 c + bab + acb - \alpha_2 b = 0.$$

Hence a direct calculations shows that

$$u^2v + uv^2 = \alpha_1 a - \alpha_1 c - bab + bac - acb + bcb = -\alpha_2 b + \alpha_2 b = 0.$$

Moreover

$$uvw = uv(-u - v) = -u^2v - uv^2 = 0.$$

Finally

$$u^3 = -u^2(v + w) = -u(uv + uw + vw) + uvw = (3\alpha_1 - \alpha_2)u.$$

This completes the proof. \square

Lemma 2.6. *Assume that $3\alpha_1 - \alpha_2 \neq 0$. Then the elements*

$$\begin{aligned} e_1 &= \frac{1}{3\alpha_1 - \alpha_2}((b + c)^2 - (\alpha_1 + \alpha_2)) = \frac{1}{3\alpha_1 - \alpha_2}(c - a)(b - a), \\ e_2 &= \frac{1}{3\alpha_1 - \alpha_2}((a + c)^2 - (\alpha_1 + \alpha_2)) = \frac{1}{3\alpha_1 - \alpha_2}(c - b)(a - b), \\ e_3 &= \frac{1}{3\alpha_1 - \alpha_2}((a + b)^2 - (\alpha_1 + \alpha_2)) = \frac{1}{3\alpha_1 - \alpha_2}(a - c)(b - c), \end{aligned}$$

form a set of orthogonal central idempotents of $\mathcal{D}_3(\alpha_1, \alpha_2)$.

Proof. Clearly $1 = e_1 + e_2 + e_3$. A direct calculation using Lemma 2.5 shows that e_2 is an idempotent that commutes with a , b and c . Then $e_1 = (12) \cdot e_2$ and $e_3 = (23) \cdot e_2$ are central idempotents. From Lemma 2.5 it now follows that e_1 , e_2 and e_3 are orthogonal. \square

Lemma 2.7. *Assume that $3\alpha_1 - \alpha_2 \neq 0$. Let V be a vector space with basis $\{x_1, x_2\}$ and let $q: V \rightarrow \mathbb{K}$ be the quadratic form given by*

$$q(\lambda_1 x_1 + \lambda_2 x_2) = \alpha_1 \lambda_1^2 + (\alpha_2 - \alpha_1) \lambda_1 \lambda_2 + \alpha_1 \lambda_2^2.$$

Then $e_3 \mathcal{D}_3(\alpha_1, \alpha_2) \simeq C(V, q)$.

Proof. Lemma 2.5 implies that $e_3(a - b) = 0$. Then the algebra $e_3 \mathcal{D}_3(\alpha_1, \alpha_2)$ with unit e_3 is given by generators $e_3 a, e_3 c$ and relations

$$(2.1) \quad (e_3 a)^2 = (e_3 c)^2 = \alpha_1 e_3,$$

$$(2.2) \quad e_3 c e_3 a + e_3 a e_3 c = (\alpha_2 - \alpha_1) e_3,$$

$$(2.3) \quad (e_3 a - e_3 c)^2 = (3\alpha_1 - \alpha_2) e_3.$$

Since (2.3) follows from the other equalities, the lemma holds. \square

Proposition 2.8. *Assume that $(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$. Then $e_1 \mathcal{D}_3(\alpha_1, \alpha_2)$, $e_2 \mathcal{D}_3(\alpha_1, \alpha_2)$ and $e_3 \mathcal{D}_3(\alpha_1, \alpha_2)$ are simple algebras isomorphic to $\mathbb{K}^{2 \times 2}$.*

Proof. Using the group action one proves that these algebras are isomorphic. So it suffices to prove that $e_3 \mathcal{D}_3(\alpha_1, \alpha_2)$ is a simple algebra. The simplicity of $e_3 \mathcal{D}_3(\alpha_1, \alpha_2)$ follows from Lemma 2.7 and Theorem 1.3. \square

Next we discuss the deformations $\mathcal{D}_3(\alpha_1, \alpha_2)$ with $3\alpha_1 - \alpha_2 = 0$.

Proposition 2.9. *Let \mathcal{L} denote the left ideal of $\mathcal{D}_3(\alpha_1, 3\alpha_1)$ generated by $a - b$ and $b - c$. Then \mathcal{L} is a two-sided nilpotent ideal of $\mathcal{D}_3(\alpha_1, 3\alpha_1)$. The quotient algebra $\mathcal{D}_3(\alpha_1, 3\alpha_1)/\mathcal{L}$ has dimension 2 and is isomorphic to the semisimple algebra $\mathbb{K}[a]/(a^2 - \alpha_1)$.*

Proof. Let $u = a - b$ and $v = b - c$. Lemma 2.5 implies that \mathcal{L} is a two-sided ideal. Moreover, $u^3 = 0$, and by acting with the transposition (13) we also obtain that $v^3 = 0$. Since $uv = vu$, it follows that \mathcal{L} is nilpotent.

Adding u and v to the defining ideal of $\mathcal{D}_3(\alpha_1, 3\alpha_1)$, one obtains the ideal

$$(b - a, c - a, a^2 - \alpha_1, 3a^2 - 3\alpha_1).$$

This implies the last claim. \square

Now we prove the main theorems of this section.

Theorem 2.10. *The algebra $\mathcal{D}_3(\alpha_1, \alpha_2)$ is semisimple if and only if*

$$(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0.$$

In this case $\mathcal{D}_3(\alpha_1, \alpha_2) \simeq (\mathbb{K}^{2 \times 2})^3$.

Proof. By Proposition 2.9, $\mathcal{D}_3(\alpha_1, 3\alpha_1)$ is not semisimple. So we may assume that $3\alpha_1 - \alpha_2 \neq 0$. We decompose $\mathcal{D}_3(\alpha_1, \alpha_2)$ as

$$\mathcal{D}_3(\alpha_1, \alpha_2) \simeq e_1 \mathcal{D}_3(\alpha_1, \alpha_2) \oplus e_2 \mathcal{D}_3(\alpha_1, \alpha_2) \oplus e_3 \mathcal{D}_3(\alpha_1, \alpha_2),$$

where the e_i are the central idempotents of Lemma 2.6. Now Proposition 2.8 implies that $\mathcal{D}_3(\alpha_1, \alpha_2)$ is semisimple and has dimension 12 if $\alpha_1 + \alpha_2 \neq 0$. In the case where $\alpha_1 + \alpha_2 = 0$, the deformation $\mathcal{D}_3(\alpha_1, \alpha_2)$ is not semisimple by Lemma 2.7 and Theorem 1.3(1). \square

Theorem 2.11. *Let $\alpha_1, \alpha_2 \in \mathbb{K}$. Then $\mathcal{D}_3(\alpha_1, \alpha_2)$ is a PBW deformation of \mathcal{E}_3 and*

$$(2.4) \quad (a-b)^{n_1} a^{n_2} c^{n_3}, \quad 0 \leq n_1 \leq 2, 0 \leq n_2, n_3 \leq 1.$$

is a basis of $\mathcal{D}_3(\alpha_1, \alpha_2)$.

Proof. Let $u = a - b$. Consider the lexicographic ordering on the words in the letters u, a and c , induced by $u < a < c$. Definition 2.2 and Lemma 2.5 imply that

$$\begin{aligned} au &= u^2 - ua, \\ cu &= u^2 - 2ua + uc, \\ ca &= \alpha_2 - \alpha_1 + u^2 - ua + uc - ac. \end{aligned}$$

Then (2.4) spans $\mathcal{D}_3(\alpha_1, \alpha_2)$. Now the corollary follows from Theorem 2.10 and Proposition 1.2. \square

Corollary 2.12. *Any polynomial identity of $(\mathbb{K}^{2 \times 2})^3$ is a polynomial identity of $\mathcal{D}_3(\alpha_1, \alpha_2)$ for any $\alpha_1, \alpha_2 \in \mathbb{K}$.*

Proof. It follows from Theorem 2.10 and an argument similar to Proposition 1.2. \square

The following result classifies PBW deformation of \mathcal{E}_3 . These deformations already appeared in [14, Theorem 6.2] and were used to obtain the classification of finite-dimensional pointed Hopf algebras with coradical isomorphic to \mathbb{S}_3 , see also [2].

Theorem 2.13. *Each PBW deformation of \mathcal{E}_3 in the category of \mathbb{S}_3 -modules is of the form $\mathcal{D}_3(\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{K}$.*

Proof. Let \mathcal{D} be a PBW deformation of \mathcal{E}_3 . Theorem 2.11 implies that (2.4) is a basis of \mathcal{D} . Since $a^2 = 0$ in \mathcal{E}_3 , there exist $\lambda_1, \dots, \lambda_4 \in \mathbb{K}$ such that

$$a^2 + \lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 = 0$$

in \mathcal{D} . Acting with the transposition (12) one obtains

$$2\lambda_1 a + (\lambda_2 + \lambda_3)(b + c) = 0$$

and hence $\lambda_1 = \lambda_2 + \lambda_3 = 0$. By acting with (13) and (23) we also obtain that

$$b^2 + \lambda_2(c - a) + \lambda_4 = 0, \quad c^2 + \lambda_2(a - b) + \lambda_4 = 0.$$

Since $ab + bc + ca = 0$ in \mathcal{E}_3 , there exist $\lambda_5, \dots, \lambda_8 \in \mathbb{K}$ such that

$$ab + bc + ca + \lambda_5 a + \lambda_6 b + \lambda_7 c + \lambda_8 = 0$$

in \mathcal{D} . By acting with the transpositions (12) and (13) we obtain that

$$ac + cb + ba - \lambda_5 a - \lambda_6 c - \lambda_7 b + \lambda_8 = 0,$$

$$cb + ba + ac - \lambda_5 c - \lambda_6 b - \lambda_7 a + \lambda_8 = 0.$$

This implies that $\lambda_5 = \lambda_6 = \lambda_7$. Hence

$$c(a^2 + \lambda_2(b - c) + \lambda_4) = (ab + bc + ca + \lambda_5(a + b + c) + \lambda_8)a$$

and therefore $\lambda_2 = \lambda_5 = 0$. Thus $\mathcal{D} = \mathcal{D}_3(-\lambda_4, -\lambda_8)$. \square

We now describe the cases where $\mathcal{D}_3(\alpha_1, \alpha_2)$ is not semisimple.

Proposition 2.14. *Assume that $\alpha_1 \neq 0$. Then the deformation $\mathcal{D}_3(\alpha_1, -\alpha_1)$ is isomorphic to the product of three copies of the preprojective algebra of the Dynkin quiver of type A_2 .*

Proof. It follows directly from Lemma 2.7. \square

Proposition 2.15. *Assume that $\alpha_1 \neq 0$. Then the deformation $\mathcal{D}_3(\alpha_1, 3\alpha_1)$ can be presented as a quiver with relations in the following way: The quiver has two vertices 1 and 2, there are two arrows u_{12}, v_{12} from 1 to 2 and two arrows u_{21}, v_{21} from 2 to 1. The relations are those of the coinvariant ring of \mathbb{S}_3 , i.e.*

$$u_{ij}v_{ji} = v_{ij}u_{ji}, \quad u_{ij}v_{ji} + v_{ij}w_{ji} + u_{ij}w_{ji} = 0, \quad u_{ij}v_{ji}w_{ij} = 0$$

for all $i, j \in \{1, 2\}$ with $i + j = 3$, where $w_{kl} = -u_{kl} - v_{kl}$ for all $k, l \in \{1, 2\}$ with $k + l = 3$.

Proof. By Theorem 2.11, $\dim \mathcal{D}_3(\alpha_1, \alpha_2) = 12$. Let

$$f_1 = \frac{\sqrt{\alpha_1} + b}{2\sqrt{\alpha_1}}, \quad f_2 = \frac{\sqrt{\alpha_1} - b}{2\sqrt{\alpha_1}}.$$

Then f_1 and f_2 are primitive idempotents. (The primitivity follows from the fact that the quotient of the deformation by the radical is 2-dimensional, see Proposition 2.9.) The action of the transposition (13) permutes both f_1, f_2 and $u = a - b, v = b - c$. Let $w = -u - v$. By Lemma 2.5, the elements u, v and w pairwise commute and

$$u + v + w = 0, \quad uv + vw + uw = 0, \quad uvw = 0.$$

Using the equations

$$f_1 u = u f_2 - \frac{u^2}{2\sqrt{\alpha_1}}, \quad f_2 v = v f_1 - \frac{v^2}{2\sqrt{\alpha_1}},$$

one can show that the elements $f_i, f_i u^2 f_i$ and $f_i v^2 f_i$ span $f_i \mathcal{D}_3(\alpha_1, 3\alpha_1) f_i$ for $i \in \{1, 2\}$, and that $u_{ij} = f_i u f_j, v_{ij} = f_i v f_j$ and $f_i u^2 v f_j$ for $i, j \in \{1, 2\}$ and $i \neq j$ span $f_i \mathcal{D}_3(\alpha_1, 3\alpha_1) f_j$. Then one shows that f_i, u_{ij} and v_{ij} , where $i, j \in \{1, 2\}$ with $i \neq j$, generate the algebra $\mathcal{D}_3(\alpha_1, 3\alpha_1)$ and satisfy the relations in the proposition. \square

3. AN INTERMEDIATE ALGEBRA

In this section we assume that \mathbb{K} is an algebraically closed field of characteristic different from 2 and 3. Let $\alpha_1, \alpha_2 \in \mathbb{K}$ and let $\beta = 3\alpha_1 - \alpha_2$.

Definition 3.1. Let $\mathcal{K}(\alpha_1, \alpha_2)$ be the associative \mathbb{K} -algebra given by generators a, b, c, y and relations

$$a^2 = b^2 = c^2 = \alpha_1, \quad ab + bc + ca = \alpha_2, \quad ac + cb + ba = \alpha_2 + y.$$

For any $\alpha_3 \in \mathbb{K}$ let $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3) = \mathcal{K}(\alpha_1, \alpha_2)/(y^3 - \alpha_3)$.

Lemma 3.2. In the algebra $\mathcal{K}(\alpha_1, \alpha_2)$ the following relations hold.

$$\begin{aligned} ya &= cy, \quad yb = ay, \quad yc = by, \\ bab - aba &= \alpha_2(b - a), \quad (b - a)^3 = \beta(b - a). \end{aligned}$$

In particular, y^3 is a central element of $\mathcal{K}(\alpha_1, \alpha_2)$.

Proof. First we obtain that

$$\begin{aligned} 0 &= b(ab + bc + ca - \alpha_2) - (ab + bc + ca - \alpha_2)a \\ &= bab + \alpha_1 c + bca - \alpha_2 b - aba - bca - \alpha_1 c + \alpha_2 a \\ &= bab - aba - \alpha_2(b - a). \end{aligned}$$

This implies directly the last equation of the proposition. Now we conclude that

$$\begin{aligned} yb - ay &= (ac + cb + ba - \alpha_2)b - a(ac + cb + ba - \alpha_2) \\ &= acb + \alpha_1 c + bab - \alpha_2 b - \alpha_1 c - acb - aba + \alpha_2 a \\ &= 0. \end{aligned}$$

The remaining two equations follow from the last one using that the cyclic group C_3 acts on $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ by permuting the generators a, b, c cyclically and by fixing y . \square

Lemma 3.3. Let $\zeta \in \mathbb{K}$ be such that $\zeta^2 + \zeta + 1 = 0$ and let

$$t = a + b + c, \quad v_+ = a + \zeta b + \zeta^2 c, \quad v_- = a + \zeta^{-1} b + \zeta^{-2} c$$

be in $\mathcal{K}(\alpha_1, \alpha_2)$. Then

$$(3.1) \quad yv_+ = \zeta v_+ y, \quad yv_- = \zeta^{-1} v_- y,$$

$$(3.2) \quad v_+ v_- = \beta + \zeta y, \quad v_- v_+ = \beta + \zeta^2 y,$$

$$(3.3) \quad tv_+ = -v_+ t - v_-^2, \quad tv_- = -v_- t - v_+^2.$$

Proof. The first two equalities follow from Lemma 3.2. The proof of the other formulas is straightforward from the definitions. \square

Lemma 3.4. Let $\zeta \in \mathbb{K}$ be such that $\zeta^2 + \zeta + 1 = 0$. Then the elements $v_+^{n_1} t^{n_2} y^{n_3}$ and $v_-^{n_1} t^{n_2} y^{n_3}$, where $n_1, n_2, n_3 \in \mathbb{N}_0$, span $\mathcal{K}(\alpha_1, \alpha_2)$.

Proof. Clearly, v_+ , v_- , t and y generate $\mathcal{K}(\alpha_1, \alpha_2)$. Since $yt = ty$, (3.1) and (3.3) imply that $\mathcal{K}(\alpha_1, \alpha_2)$ is spanned by the monomials

$$v_{s_1} v_{s_2} \cdots v_{s_r} t^{n_2} y^{n_3}, \quad r, n_2, n_3 \in \mathbb{N}_0, \quad s_1, \dots, s_r \in \{-, +\}.$$

Now use (3.2) and (3.1) to conclude the lemma. \square

Lemma 3.5. *Let $\zeta \in \mathbb{K}$ be such that $\zeta^2 + \zeta + 1 = 0$. Then the following formulas hold in $\mathcal{K}(\alpha_1, \alpha_2)$:*

$$(3.4) \quad t^2 = y + 3\alpha_1 + 2\alpha_2,$$

$$(3.5) \quad tv_+^3 + v_+^3 t = 2y^2 - 2\beta y - \beta^2,$$

$$(3.6) \quad v_+^6 = y^3 + \beta^3,$$

$$(3.7) \quad v_+^3 = v_-^3.$$

Proof. The first equality follows from the definitions. Using the last two equations in Lemma 3.3 we obtain that

$$\begin{aligned} tv_+^3 + v_+^3 t &= t(-tv_- - v_- t)v_+ + v_+(-tv_- - v_- t)t \\ &= -t^2 v_- v_+ + (v_+^2 + v_- t)tv_+ + v_+ t(v_+^2 + tv_-) - v_+ v_- t^2 \\ &= -t^2 v_- v_+ + v_- t^2 v_+ + v_+ t^2 v_- - v_+ v_- t^2 - v_+ v_-^2 v_+. \end{aligned}$$

Then (3.5) follows from (3.4) and the other equations in Lemma 3.3. Using (3.3) we obtain that

$$v_+^3 = v_+^2 v_+ = -tv_- v_+ - v_- t v_+ = -v_- v_+ t - v_- t v_+ = v_- v_-^2 = v_-^3.$$

Finally,

$$v_+^6 = v_+^3 v_-^3 = (\beta + \zeta y)(\beta + \zeta^2 y)(\beta + y) = y^3 + \beta^3$$

because of (3.7), (3.1) and (3.2). \square

Lemma 3.6. *Let $\alpha_3 \in \mathbb{K} \setminus \{0\}$. There exists $\lambda \in \mathbb{K}$ such that $v_+ + \lambda v_-^2$ is invertible in $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$.*

Proof. Assume first that $\alpha_3 + \beta^3 \neq 0$. Then v_+^6 is a non-zero constant by Equation (3.6) of Lemma 3.5 and hence $\lambda = 0$ works. Now assume that $\alpha_3 + \beta^3 = 0$. Since $y^3 = -\beta^3$, there exist orthogonal idempotents $e_0, e_1, e_2 \in \mathbb{K}[y]$ such that $ye_i = -\zeta^i \beta e_i$ for all $i \in \{0, 1, 2\}$ and $e_0 + e_1 + e_2 = 1$. Let $\lambda \in \mathbb{K} \setminus \{0\}$. Then Lemma 3.5 implies that

$$(v_+ + \lambda v_-^2)(v_- - \lambda v_+^2) = \beta + \zeta y - \lambda^2(\beta + \zeta^2 y)(\beta + y)$$

Then

$$e_i(v_+ + \lambda v_-^2)(v_- - \lambda v_+^2) = e_i \beta (1 - \zeta^{i+1} - \lambda^2 \beta (1 - \zeta^i)(1 - \zeta^{2+i}))$$

is non-zero for all $i \in \{0, 1, 2\}$. Hence $v_+ + \lambda v_-^2$ is invertible. \square

For the formulation of the next claim we need additional notation. For any $\gamma \in \mathbb{K}$ let (V, q_γ) be a two-dimensional quadratic space with basis x_1, x_2 such that

$q_\gamma(\lambda_1 x_1 + \lambda_2 x_2) = (\gamma + 3\alpha_1 + 2\alpha_2)\lambda_1^2 + (2\gamma^2 - 2\beta\gamma - \beta^2)\lambda_1\lambda_2 + (\gamma^3 + \beta^3)\lambda_2^2$ for all $\lambda_1, \lambda_2 \in \mathbb{K}$. If $\gamma^3 + \beta^3 \neq 0$, then let $x', x'' \in C(V, q_\gamma)$ be the elements

$$(3.8) \quad \begin{aligned} x' &= -x_1 - (\beta + \zeta^2\gamma)^{-1}x_2, \\ x'' &= x_1 + (\beta + \zeta^2\gamma)^{-1}x_2 - (\beta + \gamma)^{-1}x_2 \end{aligned}$$

and let $Y, A, B, C \in C(V, q_\gamma)^{3 \times 3}$ be the matrices

$$\begin{aligned} Y &= \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \zeta\gamma & 0 \\ 0 & 0 & \zeta^2\gamma \end{pmatrix}, \\ A &= \frac{1}{3} \begin{pmatrix} x_1 & (\beta + \zeta^2\gamma) & x_2 \\ 1 & x' & \beta + \gamma \\ (\beta + \zeta\gamma)x_2^{-1} & 1 & x'' \end{pmatrix}, \\ B &= \frac{1}{3} \begin{pmatrix} x_1 & \zeta(\beta + \zeta^2\gamma) & \zeta^2x_2 \\ \zeta^2 & x' & \zeta(\beta + \gamma) \\ \zeta(\beta + \zeta\gamma)x_2^{-1} & \zeta^2 & x'' \end{pmatrix}, \\ C &= \frac{1}{3} \begin{pmatrix} x_1 & \zeta^2(\beta + \zeta^2\gamma) & \zeta x_2 \\ \zeta & x' & \zeta^2(\beta + \gamma) \\ \zeta^2(\beta + \zeta\gamma)x_2^{-1} & \zeta & x'' \end{pmatrix}. \end{aligned}$$

Remark 3.7. The discriminant of the quadratic form q_γ above is

$$-9(4\alpha_1\gamma^3 + \beta^3(\alpha_1 + \alpha_2)).$$

Thus the quadratic space (V, q_γ) is nondegenerate if and only if this expression is non-zero.

Lemma 3.8. *Let $\gamma, \zeta \in \mathbb{K}$ with $\zeta^2 + \zeta + 1 = 0$. Assume that $\gamma^3 + \beta^3 \neq 0$. Then there exists an algebra map $\rho_\gamma : \mathcal{K}(\alpha_1, \alpha_2) \rightarrow C(V, q_\gamma)^{3 \times 3}$ such that*

$$\rho_\gamma(a) = A, \quad \rho_\gamma(b) = B, \quad \rho_\gamma(c) = C, \quad \rho_\gamma(y) = Y.$$

This map also satisfies the identity $\rho_\gamma(y^3) = \gamma^3 \text{id}$.

Proof. The matrices A, B, C, Y are well-defined. It is straightforward to check that the equations

$$A^2 = B^2 = C^2 = \alpha_1 \text{id},$$

$$AB + BC + CA = \alpha_2 \text{id}, \quad AC + CB + BA = \alpha_2 \text{id} + Y, \quad Y^3 = \gamma^3 \text{id}$$

hold. \square

Proposition 3.9. *Let $\gamma, \zeta \in \mathbb{K}$ such that $\zeta^2 + \zeta + 1 = 0$ and $\gamma(\gamma^3 + \beta^3) \neq 0$. Let M be a simple $C(V, q_\gamma)$ -module. Then M^3 is a simple $\mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$ -module via*

$$xm = \rho_\gamma(x)m \quad \text{for all } x \in \mathcal{K}(\alpha_1, \alpha_2, \gamma^3), m \in M^3.$$

Proof. Lemma 3.8 implies that M^3 is a left $\mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$ -module. Let N be a non-zero submodule of M^3 . Since \mathbb{K} is algebraically closed, there exists an eigenvector $m \in N$ of $\rho_\gamma(y)$. Moreover, Equation (3.6) implies that $\rho_\gamma(v_+^6) = (\gamma^3 + \beta^3)\text{id} \neq 0$. Thus, by (3.1) we may assume that $ym = \gamma m$. Since $\gamma \neq 0$, we conclude that $m \in M \times \{0\} \times \{0\}$. Now observe that

$$x_1 m' = \rho_\gamma(t) m', \quad x_2 m' = \rho_\gamma(v_+^3) m' \quad \text{for all } m' \in M \times \{0\} \times \{0\}.$$

Hence the simplicity of M implies that $M \times \{0\} \times \{0\} \subseteq N$. Then $N = M$ because of (3.1). \square

Theorem 3.10. *Let $\alpha_3 \in \mathbb{K}$.*

(1) *The elements*

$$(a - b)^{n_1} a^{n_2} c^{n_3} y^{n_4}, \quad n_1, n_4 \in \{0, 1, 2\}, \quad n_2, n_3 \in \{0, 1\},$$

form a basis of $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$. In particular, $\dim \mathcal{K}(\alpha_1, \alpha_2, \alpha_3) = 36$.

(2) *$\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ is a PBW deformation of $\mathcal{K}(0, 0, 0)$.*

Proof. Similarly to the proof of Theorem 2.11 one shows that the elements in (1) span $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$. In particular, $\dim \mathcal{K}(\alpha_1, \alpha_2, \alpha_3) \leq 36$.

Let $\gamma \in \mathbb{K}$ with $\gamma^3 = \alpha_3$. Assume that

$$(3.9) \quad \gamma(\gamma^3 + \beta^3)(4\alpha_1\gamma^3 + \beta^3(\alpha_1 + \alpha_2)) \neq 0.$$

By Remark 3.7, the quadratic space (V, q_γ) is nondegenerate. Thus $C(V, q_\gamma)$ is simple by Theorem 1.3(2). Hence there exists a 2-dimensional simple $C(V, q_\gamma)$ -module. Since $\gamma(\gamma^3 + \beta^3) \neq 0$, by Proposition 3.9 there exists a simple 6-dimensional $\mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$ -module. Hence $\dim \mathcal{K}(\alpha_1, \alpha_2, \gamma^3) = 36$. The variety of all triples $(\alpha_1, \alpha_2, \gamma^3)$ satisfying (3.9) is a dense subvariety of \mathbb{K}^3 . Therefore the theorem follows from Proposition 1.2. \square

In order to determine the radical of the deformation $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ the following proposition is useful.

Proposition 3.11. *Let A be a \mathbb{K} -algebra and $x, y \in A$. Let $n \in \mathbb{N}$ and $\zeta, \lambda \in \mathbb{K} \setminus \{0\}$. Assume that $\text{char } \mathbb{K}$ does not divide n , ζ is a primitive root of 1 of order n , x is invertible, $yx = \zeta xy$, and $y^n = \lambda$. Then there exists a primitive idempotent $e \in \mathbb{K}[y]$ such that the following claims hold.*

- (1) $A = \bigoplus_{i,j=0}^{n-1} x^i e A e x^j$.
- (2) *The map given by $I \mapsto e I e$ is a bijection between the ideals of A and the ideals of $e A e$. The inverse map is given by $e I e \mapsto A e I e A$.*
- (3) $\text{Rad } A = \bigoplus_{i,j=0}^{n-1} x^i \text{Rad}(e A e) x^j$.

Proof. Let $\gamma \in \mathbb{K}$ be such that $\gamma^n = \lambda$. Since $\text{char } \mathbb{K} \nmid n$ and $y^n = \lambda$, there exist unique idempotents $e = e_0, e_1, \dots, e_{n-1} \in \mathbb{K}[y]$ such that $y e_i = \gamma \zeta^i e_i$ for all i . Moreover, these idempotents are orthogonal and primitive. Using these properties, one checks that $e_i = x^{i+kn} e x^{-i-kn}$ for all i and all $k \in \mathbb{Z}$.

- (1) Use that $A = \bigoplus_{i,j=0}^{n-1} e_i A e_j$ and that $e_i = x^i e x^{-i}$ for all i .

(2) One has to check that $I = AeIeA$ for all ideals I of A and that $eAeJeAe = eJe$ for all ideals eJe of eAe . The second claim is obvious. The first one follows from

$$I = \sum_{i,j=0}^{n-1} e_i I e_j = \sum_{i,j=0}^{n-1} x^i e x^{-i} I x^j e x^{-j} \subseteq AeIeA.$$

(3) Since $\text{Rad}(eAe) = e(\text{Rad } A)e$, (2) implies that

$$\text{Rad } A = Ae(\text{Rad } A)eA = \bigoplus_{i,j=0}^{n-1} e_i Ae \text{Rad}(eAe) eAe_j.$$

Now use that $e_i = x^i e x^{-i}$ and $e_j = x^{j-n} e x^{n-j}$ for all $i, j \in \{1, \dots, n-1\}$ and that $\text{Rad}(eAe)$ is an ideal of eAe . \square

Lemma 3.12. *Let $\gamma \in \mathbb{K} \setminus \{0\}$ and $e \in \mathbb{K}[y] \subseteq \mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$ be an idempotent. Assume that $ye = \gamma e$. Then the algebra $e\mathcal{K}(\alpha_1, \alpha_2, \gamma^3)e$ is generated by et and ev_+^3 and is isomorphic to the Clifford algebra $C(V, q_\gamma)$.*

Proof. Let $A = \mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$ and let $\zeta \in \mathbb{K}$ be such that $\zeta^2 + \zeta + 1 = 0$. Then A is generated by t, v_+, v_- and y . Lemma 3.3 implies that $v_+^{n_1} t^{n_2} y^{n_3}$, $v_-^{n_1} t^{n_2} y^{n_3}$, where $n_1, n_2, n_3 \in \mathbb{N}_0$, span A .

By Lemma 3.6, there exists $\lambda \in \mathbb{K}$ such that $x = v_+ + \lambda v_-^2 \in A$ is invertible. Moreover, $yx = \zeta xy$ by Lemma 3.3. Then Proposition 3.11 implies that $A = \bigoplus_{i,j=0}^2 x^i e A e x^j$. Note that $v_+^{n_1} t^{n_2} y^{n_3} e \in x^{n'_1} e A e$ and $v_-^{m_1} t^{n_2} y^{n_3} e \in x^{m'_1} e A e$ for all $n_1, m_1, n_2, n_3 \in \mathbb{N}_0$, where $n'_1, m'_1 \in \{0, 1, 2\}$ such that $n_1 \equiv n'_1 \pmod{3}$ and $m_1 \equiv -m'_1 \pmod{3}$. Therefore eAe is generated by $ey = \gamma e$, et and $ev_+^3 = ev_-^3$, see (3.7). Moreover, Theorem 3.10(1) and Proposition 3.11(1) imply that $\dim eAe = 4$. Then the claim follows from Lemma 3.5. \square

Corollary 3.13. *Let $\gamma \in \mathbb{K}$.*

- (1) *If $\gamma = 0$ then y generates a nilpotent ideal of $\mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$.*
- (2) *If $\gamma \neq 0$ then $\mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$ is semisimple if and only if q_γ is nondegenerate. In this case, $\mathcal{K}(\alpha_1, \alpha_2, \gamma^3) \simeq \mathbb{K}^{6 \times 6}$.*

Proof. Let $A = \mathcal{K}(\alpha_1, \alpha_2, \gamma^3)$.

(1) Lemma 3.2 implies that $N = Ay$ is a two-sided ideal of A . If $\gamma = 0$, then $y^3 = 0$ in A and hence $N^3 = 0$.

(2) Assume that $\gamma \neq 0$. Let $\zeta \in \mathbb{K}$ be such that $\zeta^2 + \zeta + 1 = 0$. By Lemma 3.6, there exists an invertible element $x \in A$ such that $yx = \zeta xy$. According to Proposition 3.11, there exists a primitive idempotent $e \in \mathbb{K}[y]$ such that $\text{Rad } A = \bigoplus_{i,j=0}^2 x^i \text{Rad}(eAe) x^j$. Let $\gamma \in \mathbb{K} \setminus \{0\}$ be such that $ye = \gamma e$. Then $eAe \simeq C(V, q_\gamma)$ by Lemma 3.12. Thus the claim on the semisimplicity of A follows from Theorem 1.3.

Assume that q_γ is nondegenerate. Then $C(V, q_\gamma)$ is simple by Theorem 1.3(2). Hence A is simple by Proposition 3.11(2). Since $\dim A = 36$ by Theorem 3.10(1), we conclude that $A \simeq \mathbb{K}^{6 \times 6}$. \square

4. THE NICHOLS ALGEBRA OF DIMENSION 72

Again we assume that \mathbb{K} is an algebraically closed field of characteristic different from 2 and 3. In this section we study the algebra B presented by generators a, b, c, d and relations

$$\begin{aligned} a^2 &= b^2 = c^2 = d^2 = 0, \\ ab + bc + ca &= ac + cd + da = ad + ba + db = bd + cb + dc = 0, \\ (a + b + c)^6 &= 0. \end{aligned}$$

Based on computer calculations it is known that $\dim B = 72$ and that the Hilbert series of B is

$$H(t) = 1 + 4t + 8t^2 + 11t^3 + 12t^4 + 12t^5 + 11t^6 + 8t^7 + 4t^8 + t^9.$$

In Theorem 4.9 we will prove these facts by different methods.

Definition 4.1. For any $\alpha_1, \alpha_2 \in \mathbb{K}$, let $\mathcal{T}(\alpha_1, \alpha_2)$ be the \mathbb{K} -algebra given by generators a, b, c, d and relations

$$\begin{aligned} a^2 - \alpha_1 &= b^2 - \alpha_1 = c^2 - \alpha_1 = d^2 - \alpha_1 = 0, \\ ca + bc + ab - \alpha_2 &= da + cd + ac - \alpha_2 = 0, \\ db + ba + ad - \alpha_2 &= dc + cb + bd - \alpha_2 = 0. \end{aligned}$$

Let $y = ac + cb + ba - \alpha_2$.

Remark 4.2. Let G be the group given by generators g_a, g_b, g_c, g_d with relations

$$\begin{aligned} g_a g_b &= g_b g_c = g_c g_a, & g_a g_c &= g_c g_d = g_d g_a, \\ g_a g_d &= g_d g_b = g_b g_a, & g_b g_d &= g_d g_c = g_c g_b. \end{aligned}$$

It is known that G is a central extension of $\mathrm{SL}(2, 3)$.

There is a unique $\mathbb{K}G$ -module algebra structure on $\mathcal{T}(\alpha_1, \alpha_2)$ such that g_a, g_b, g_c, g_d act on the generators a, b, c, d according to Table 4.1.

TABLE 4.1. The action of G .

	a	b	c	d
g_a	$-a$	$-c$	$-d$	$-b$
g_b	$-d$	$-b$	$-a$	$-c$
g_c	$-b$	$-d$	$-c$	$-a$
g_d	$-c$	$-a$	$-b$	$-d$

Remark 4.3. The usual presentation for B found in the literature involves a different degree-six relation. One computes

$$(cb)^2 = c(bc)b = c(-ab - ca)b = 0.$$

Then acting on this with g_d and g_d^2 one obtains $(ac)^2 = (ba)^2 = 0$. Now a direct computation shows that

$$(cb + ba + ac)^3 = (cba)^2 + (bac)^2 + (acb)^2.$$

Remark 4.4. The algebra B is the Nichols algebra associated with the rack given by a conjugacy class of 3-cycles in the alternating group \mathbb{A}_4 and constant cocycle -1 . It was found by Graña in [15] and later used by Ngakeu, Majid and Lambert in noncommutative geometry [22]. The group G of Remark 4.2 is the enveloping group of this rack.

Lemma 4.5. *Let $\alpha_1, \alpha_2 \in \mathbb{K}$. Then $yx = -(g_d \cdot x)y$ in $\mathcal{T}(\alpha_1, \alpha_2)$ for all $x \in \{a, b, c, d\}$.*

Proof. Let $X = \{a, b, c, d\}$. The enveloping group permutes the elements y_i , $i \in X$. The set $X \times X$ has two orbits with respect to the action of G . One of them contains (a, a) and the other (a, b) . Therefore it suffices to prove that

$$yd = dy, \quad yb = ay.$$

Equation $yd = dy$ is proved as follows.

$$\begin{aligned} yd &= (ba + ac + cb - \alpha_2)d \\ &= b(\alpha_2 - db - ba) + a(\alpha_2 - da - ac) + c(\alpha_2 - dc - cb) - \alpha_2d \\ &= (\alpha_2 - bd)b + (\alpha_2 - ad)a + (\alpha_2 - cd)c - \alpha_2d - \alpha_1(a + b + c) \\ &= (dc + cb)b + (db + ba)a + (da + ac)c - \alpha_2d - \alpha_1(a + b + c) \\ &= dy. \end{aligned}$$

Equation $yb = cy$ is obtained by the following steps:

$$\begin{aligned} yb &= (ba + ac + cb - \alpha_2)b \\ &= b(ab - \alpha_2) + acb + \alpha_1c \\ &= b(-ca - bc) + a(cb + ac) \\ &= -\alpha_1c - (\alpha_2 - ca - ab)a + a(cb + ac) \\ &= a(-\alpha_2 + ba + cb + ac) \\ &= ay. \end{aligned}$$

This completes the proof. \square

Definition 4.6. *For any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$, let $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ be the deformation of B given by generators a, b, c, d and relations*

$$\begin{aligned} a^2 - \alpha_1 &= b^2 - \alpha_1 = c^2 - \alpha_1 = d^2 - \alpha_1 = 0, \\ ca + bc + ab - \alpha_2 &= da + cd + ac - \alpha_2 = 0, \\ db + ba + ad - \alpha_2 &= dc + cb + bd - \alpha_2 = 0, \\ (cb + ba + ac - \alpha_2)^3 - \alpha_3 &= 0. \end{aligned}$$

Remark 4.7. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$. The algebra $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is naturally a $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ -bimodule where the action of a, b, c and y is given by multiplication with a, b, c and y , respectively.

Proposition 4.8. *The algebra $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is isomorphic to the Ore extension $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)[d; \partial, \sigma]/(d^2 - \alpha_1)$, where*

$$\sigma \in \text{Aut}(\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)), \quad \sigma(a) = -c, \quad \sigma(b) = -a, \quad \sigma(c) = -b,$$

and the (σ, id) -skew derivation ∂ of $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\partial(a) = \alpha_2 - ac, \quad \partial(b) = \alpha_2 - ba, \quad \partial(c) = \alpha_2 - cb.$$

Proof. It is straightforward to prove that the automorphism σ and the skew derivation ∂ of $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ exist. For example:

$$\partial(a^2) = \partial(a)a + \sigma(a)\partial(a) = (\alpha_2 - ac)a - c(\alpha_2 - ac) = 0.$$

The rest follows from the definitions of $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ and $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$. \square

Theorem 4.9. *For any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$ the algebra $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is a PBW deformation of B and*

$$(b - a)^{n_1} a^{n_2} c^{n_3} y^{n_4} d^{n_5}, \quad n_1, n_4 \in \{0, 1, 2\}, n_2, n_3, n_5 \in \{0, 1\},$$

is a basis of $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$.

Proof. First one checks that $(d^2 - \alpha_1)a = b(d^2 - \alpha_1)$. By acting on this equation with $g_d \in G$ it follows that the left ideal of $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ generated by $d^2 - \alpha_1$ is a two-sided ideal. Hence

$$(4.1) \quad \mathcal{T}(\alpha_1, \alpha_2, \alpha_3) \simeq \mathcal{K}(\alpha_1, \alpha_2, \alpha_3) \otimes \mathbb{K}[d]/(d^2 - \alpha_1)$$

as a left module over $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3)$ by Proposition 4.8, see Remark 4.7. Now apply Theorem 3.10(1) to obtain the claimed basis of $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$. Hence $\dim \mathcal{T}(\alpha_1, \alpha_2, \alpha_3) = \dim B = 72$ for all $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$. Therefore $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is a PBW deformation of B . \square

Theorem 4.10. *The algebra $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is semisimple if and only if*

$$\alpha_3(\alpha_3 + (\alpha_1 + \alpha_2)(3\alpha_1 - \alpha_2)^2) \neq 0.$$

In this case $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3) \simeq (\mathbb{K}^{6 \times 6})^2$.

Proof. Assume that $\alpha_3 = 0$. Lemma 4.5 implies that the left ideal generated by y is a two-sided nilpotent ideal. Hence $\mathcal{T}(\alpha_1, \alpha_2, 0)$ is not semisimple.

Now assume that $\alpha_3 \neq 0$. Let $\zeta \in \mathbb{K}$ be such that $\zeta^2 + \zeta + 1 = 0$. Lemma 3.6 and (3.1) imply that there exists an invertible element $x \in \mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ such that $yx = \zeta xy$. By Proposition 3.11(3), there exists a primitive idempotent $e \in \mathbb{K}[y]$ such that $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is semisimple if and only if $T = e\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)e$ is semisimple. Since $y^3 = \alpha_3$, there exists $\gamma \in \mathbb{K}$ such that $ye = \gamma e$ and $\gamma^3 = \alpha_3$.

Recall that $yd = dy$ by Lemma 4.5. Therefore T is generated by et , ev_+^3 and ed by (4.1) and Lemma 3.12, where $t = a + b + c$ and $v_+ = a + \zeta b + \zeta^2 c$. Moreover, $\dim T = 8$. Now we obtain that

$$(4.2) \quad d^2 = \alpha_1, \quad dt + td = 2\alpha_2 - y, \quad dv_+^3 + v_+^3 d = 2y^2 + (3\alpha_1 - \alpha_2)^2.$$

Hence T is isomorphic to the Clifford algebra $C(W, q'_\gamma)$, where W is a three-dimensional vector space and q'_γ is the quadratic form on W given by

$$\begin{aligned} q'_\gamma(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) &= (\gamma + 3\alpha_1 + 2\alpha_2)\lambda_1^2 + (\gamma^3 + \beta^3)\lambda_2^2 + \alpha_1\lambda_3^2 \\ &\quad + (2\gamma^2 - 2\beta\gamma - \beta^2)\lambda_1\lambda_2 + (2\alpha_2 - \gamma)\lambda_1\lambda_3 + (2\gamma^2 + \beta^2)\lambda_2\lambda_3 \end{aligned}$$

with respect to a fixed basis x_1, x_2, x_3 of W , and $\beta = 3\alpha_1 - \alpha_2$. The semisimplicity of $C(W, q'_\gamma)$ is equivalent to the nondegeneracy of q'_γ , that is, to $\alpha_3 + (\alpha_1 + \alpha_2)(3\alpha_1 - \alpha_2)^2 \neq 0$.

Assume now that $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ (equivalently, T) is semisimple. By Proposition 3.11(2) and Theorem 1.3(3), the algebra $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ is the direct product of two simple ideals. Then the claim follows from the fact that $72 = 6^2 + 6^2$ is the only decomposition of 72 as a sum of two squares. \square

A result analogous to Corollary 2.12 is the following:

Corollary 4.11. *Any polynomial identity of $(\mathbb{K}^{6 \times 6})^2$ is a polynomial identity of $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$ for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$.*

Recall the definition of the group G in Remark 4.2. The following result classifies PBW deformation of the algebra B . These deformations already appeared in [14, Theorem 6.3] in the context of the classification of finite-dimensional pointed Hopf algebras with non-abelian coradical.

Theorem 4.12. *Each PBW deformation of B in the category of G -modules is of the form $\mathcal{T}(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$.*

Proof. Let T be a PBW deformation of B in the category of G -modules. Theorem 4.9 for B implies that T is given by generators a, b, c, d and relations

$$\begin{aligned} a^2 &= t_1, & b^2 &= t_2, & c^2 &= t_3, & d^2 &= t_4, \\ ca + bc + ab &= t_5, & da + cd + ac &= t_6, \\ db + ba + ad &= t_7, & dc + cb + bd &= t_8, \\ (cb + ba + ac)^3 &= t_9, \end{aligned}$$

where t_1, t_2, \dots, t_8 are linear combinations of $1, a, b, c, d$, and t_9 is a linear combination of monomials of degree at most 5 in the generators a, b, c, d . By acting with g_a^3 on the equation $a^2 = t_1$ one obtains that $a^2 = \alpha_1$ for some $\alpha \in \mathbb{K}$. By acting with g_b and g_c we conclude that $b^2 = c^2 = d^2 = \alpha_1$. Similarly, the actions of g_d^3 and g_b, g_c on the equation $ca + bc + ab = t_5$ imply that $t_5 = t_6 = t_7 = t_8 = \alpha_2$ for some $\alpha_2 \in \mathbb{K}$. Let us rewrite the last defining relation of T to be

$$y^3 = t'_9,$$

where $y = cb + ba + ac - \alpha_2$. It remains to show that $t'_9 \in \mathbb{K}1$.

Lemma 3.4 and (4.1) imply that

$$t'_9 = \sum_{n_1, \dots, n_4 \in \mathbb{N}_0} \lambda_{n_1, \dots, n_4} v_+^{n_1} t^{n_2} y^{n_3} d^{n_4} + \sum_{n_1, \dots, n_4 \in \mathbb{N}_0} \mu_{n_1, \dots, n_4} v_-^{n_1} t^{n_2} y^{n_3} d^{n_4},$$

Since $g_d \cdot y^3 = y^3$ and

$$\begin{aligned} g_d \cdot v_+^{n_1} t^{n_2} y^{n_3} d^{n_4} &= \zeta^{n_1} (-1)^{n_1 + n_2 + n_4} v_+^{n_1} t^{n_2} y^{n_3} d^{n_4}, \\ g_d \cdot v_-^{n_1} t^{n_2} y^{n_3} d^{n_4} &= \zeta^{-n_1} (-1)^{n_1 + n_2 + n_4} v_-^{n_1} t^{n_2} y^{n_3} d^{n_4}, \end{aligned}$$

we conclude that $\lambda_{n_1, \dots, n_4} = \mu_{n_1, \dots, n_4} = 0$ whenever $3 \nmid n_1$ or $n_1 + n_2 + n_4$ is odd. Moreover the degree of t'_9 is at most five. Hence $d^2 = \alpha_1$ and (3.4) imply that

$$t'_9 = \lambda_1 + \lambda_2 td + \lambda_3 y + \lambda_4 v_+^3 t + \lambda_5 v_+^3 d + \lambda_6 tyd + \lambda_7 y^2$$

for some $\lambda_1, \dots, \lambda_7 \in \mathbb{K}$. Equations $dt = -td + 2\alpha_2 - y$ and (3.5) imply that

$$t'_9 t - tt'_9 = ((\beta^2 \lambda_5 - 2\lambda_2 t^2) + 2(\beta \lambda_5 - \lambda_6 t^2)y - 2\lambda_5 y^2) d + t'',$$

for some $t'' \in \mathcal{K}(\alpha_1, \alpha_2)$. Since $y^3 t = ty^3$, using (4.1) and Theorem 4.9 we conclude that $\lambda_5 = 0$. Similarly, using (4.2) and $dy = yd$, we obtain that

$$t'_9 d - dt'_9 = (\lambda_6 y^2 + (\lambda_2 - 2\alpha_2 \lambda_6)y - 2\alpha_2 \lambda_2)d + t'''$$

for some $t''' \in \mathcal{K}(\alpha_1, \alpha_2)$. Since $y^3 d = dy^3$, we conclude from Theorem 4.9 that $\lambda_2 = \lambda_6 = 0$. Since

$$0 = t'_9(t + 2d) - (t + 2d)t'_9 = -6\lambda_4 ty^2 + \text{terms of degree } \leq 3,$$

Theorem 4.9 implies that $\lambda_4 = 0$. Finally $y^3 a = ay^3$ and Lemma 3.2 imply that

$$0 = t'_9 a - at'_9 = \lambda_3(c - a)y + \lambda_7(b - a)y^2$$

and hence $t'_9 = \lambda_1$ by Theorem 4.9. \square

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I. HECKENBERGER: PHILIPPS-UNIVERSITÄT MARBURG, FB MATHEMATIK UND INFORMATIK, HANS-MEERWEIN-STRASSE, 35032 MARBURG, GERMANY.
E-mail address: heckenberger@mathematik.uni-marburg.de

L. VENDRAMIN: IMAS–CONICET AND DEPTO. DE MATEMÁTICA, FCEN, UNIVERSIDAD DE BUENOS AIRES, PABELLÓN 1, CIUDAD UNIVERSITARIA (1428), BUENOS AIRES, ARGENTINA.
E-mail address: lvendramin@dm.uba.ar